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NOTES ON THE THEORY OF NUMBERS.

By PROFESSOR L. E. DICKSON, The University of Chicago.

1. *Even perfect numbers.* Let $\sigma(k)$ denote the sum of all the divisors of k. A perfect number k is one for which $\sigma(k)=2k$. Let $k=2^n q$, where q is odd and $n>0$. Then $(2^{n+1}-1)\sigma(q)=2^{n+1}q$. Hence $\sigma(q)=q+d$, where $d=q/(2^{n+1}-1)$. Thus d is an integral divisor of q, so that q and d are the only divisors of q. Hence $d=1$ and q is a prime. Hence every even perfect number is of Euclid's type $2^n(2^{n+1}-1)$, where the second factor is a prime. This is much simpler than any proof that has been published hitherto.

2. *Amicable numbers.* In his elaborate memoir on amicable numbers, Euler* was led (§ 95) to the type $16pq$, $16.17.137r$, where p, q, r are distinct primes. These two numbers are amicable if and only if

$$p=m+9935,\quad q=n+9935,\quad r=4(m+n)+88799,\quad mn=2^7.3^4.7.23.73.$$

Since r always exceeds 100,000, the limit of the table of primes accessible to Euler, he made no discussion of this type. In view of the large number of pairs of factors of mn (only distinct even values of m, n need be examined), it seemed likely that there exist amicable numbers of this type. A complete examination of the 120 cases showed that p, q, r are all primes only when

$$m=2^4.3^3.7,\quad p=12959,\quad q=50231,\quad r=262079;$$
$$m=2^3.3.7,\quad p=10103,\quad q=735263,\quad r=2990783.$$

The cases in which numbers exceeding 10 million had to be considered are $m=6$, $r\equiv 0 \pmod 7$; $m=32$, $r\equiv 0 \pmod 5$. The two new pairs of amicable numbers are thus:

$$16.12959.50231,\quad 16.17.137.262079;\quad 16.10103.735263,\quad 16.17.137.2990783.$$

These were checked to be amicable and the primes rechecked by Lehmer's table.

3. *Bernouillian numbers* are most conveniently employed in the symbolic notation of Lucas; we have $(b+1)^n-b^n=0$. In this notation, the generalization by Stern (Munich Akad., 1877, p. 157) of Seidel's recursion formula may be written:

$$\left\{\binom{n}{1}+\binom{m}{1}\right\} b^{m+n-1} + \left\{\binom{n}{2}-\binom{m}{2}\right\} b^{m+n-2} + \left\{\binom{n}{3}\right.$$

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* *Opuscula varii argum.*, 2, 1750, p. 23; *Commentationes Arithmeticae Collectae*, 1849, pp. 102-145.

$$+\binom{m}{3}\Big\}\, b^{m+n-3}+\ldots+\Big\{\binom{n}{n}+(-1)^{n-1}\binom{m}{n}\Big\}\, b^{m}=0,$$

for $n \geqq m$, where the binomial coefficient $\binom{m}{k}$ is zero if $k>m$. Summing separately the first terms and the second terms, we get

$$b^{m}(b+1)^{n}-b^{n}(b-1)^{m}=0.$$

To give a direct proof of the latter, we note that by definition

$$f(b+1)=f(b)+f'(0),$$

for any polynomial $f(x)$. Taking $f(x)=x^{n}(x-1)^{m}$, $n>1$, we obtain the desired formula.

Setting $\beta^{m}=2(2^{m}-1)b^{m}$, we may similarly write the other formula of Seidel and Stern as follows:

$$\beta^{m}(\beta+1)^{n}+\beta^{n}(\beta-1)^{m}=0, \quad n \geqq m>1.$$

4. $A^{3}+B^{3}+C^{3}=D^{3}$. Euler's general solution (*Comm. Arith.* 1, p. 199) is

$$A=n(gf-e^{2}), \quad B=n(hf+e^{2}), \quad C=n(f^{2}-ge), \quad D=n(f^{2}+he),$$

where we have employed the abbreviations

$$e=a^{2}+3b^{2}, \quad f=d^{2}+3c^{2}, \quad g=3ac+3bc-ad+3bd, \quad h=3ac-3bc+ad+3bd.$$

These abbreviations enable us to point out the identity which underlies his solution. In

$$\begin{aligned} A^{3}+B^{3}+C^{3}-D^{3}&=n^{3}(f^{3}-e^{3})[g^{3}+h^{3}-3ef(g+h)] \\ &=n^{3}(f^{3}-e^{3})(g+h)(g^{2}-gh+h^{2}-3ef), \end{aligned}$$

it is the final factor which vanishes. This follows from the identity

$$ef \equiv (ad-3bc)^{2}+3(ac+bd)^{2}=\left(\frac{h-g}{2}\right)^{2}+\left(\frac{h+g}{6}\right)^{2},$$

which in turn follows from $(a+b\sqrt{-3})(d+c\sqrt{-3})=ad-3bc+(ac+bd)\sqrt{-3}$.

We may also make use of the factorizations (where $\omega^3=1$):

$$A+C=(f-e)(g+f+e),\quad A+\omega C=(f-\omega e)(g+\omega f+\omega^2e),$$
$$A+\omega^2C=(f-\omega^2e)(g+\omega^2f+\omega e),$$
$$D-B=(f-e)(-h+f+e),\quad \omega D-B=(f-\omega e)(-h+\omega f+\omega^2e),$$
$$\omega^2D-B=(f-\omega^2e)(-h+\omega^2f+\omega e).$$

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

352. Proposed by E. B. ESCOTT, University of Michigan, Ann Arbor, Mich.

Solve the equations, $x^3=-8y+24\ldots(1)$.
$y^3=-8x+24\ldots(2)$.

Solution by S. LEFSCHETZ, Clark University.

The curves represented by (1) and (2) have nine common points of which three are on the line $x-y=0$, since they are evidently symmetric with respect to it. For these points, then, $x^3+8x-24=0$.

One solution is $x=y=2$; dividing by $x-2$ we obtain, $x^2+2x+12=0$. $\therefore x=y=-1+i\sqrt{11}$, and $x=y=-1-i\sqrt{11}$. To find the solutions, six in number, for which $x\neq y$, we multiply (1) by x, (2) by y, subtract and divide by $x-y$, so that we have,

$$(x+y)(x^2+y^2)=24\ldots(3).$$

Also subtracting (2) from (1), and dividing by $(x-y)$, we have

$$x^2+y^2=8-xy\ldots(4),$$

so that (3) can be written,

$$(x+y)(8-xy)=24\ldots(5),$$

which can also be written,

$$xy(x+y)=8(x+y)-24,$$